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A Note on Circular Distance

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Abstract

For any two vertices r and s in a connected graph H, if the sum of geodesic distance and detour distance between two vertices r and s is called a *circular distance*. In a class of connected graphs, it has been observed that circular diameter is twice the circular radius and equal to the circular radius. Further it has been observed that the vertex set of very graph is the vertex-to-vertex circular center of some connected graph. It has been proved that in any graph there is a block which contains the circular center. Some results have been given on trees.

Keywords: Circular eccentricity, Circular radius, Circular diameter, Pendent vertex.

1. Introduction

In this article, we consider, finite connected graph H having no loops and multiple edges. Graph distance is expansive field within graph theory with myriad practical and scientific implications. Various types of graph distances are documented in the literature.The distance, denoted as $d(r, s)$, between any two vertices r and s within graph H is determined by the shortest path length fron r to s in H. The *eccentricity* $e(r)$ of a vertex r in graph H is the farthest distance from r to any other vertex within H. The graph's radius $r(H)$ corresponds to the minimum eccentricity among its vertices, whereas the diameter $d(H)$ of H represents the maximum eccentricity observed among its vertices. The distance between two vertices constitutes a foundational concept within pure graph theory, and this measure serves as a metric on the vertex set of vertices in H. Additional findings pertaining to this distance can be discovered in the literature. Refs.[1,2,3,4,5,6].

The detour distance, characterized as the length of the longest path between two vertices in a graph, also serves as a metric on the vertex set of H in Refs[7,8].For any two vertices r and s in H, the *detour distance* $D(r, s)$ between any two vertices r and s in graph H is defined as the length of the longest path from r to s within H. The *detour eccentricity* $e_D(r)$ for a vertex r in grap H is defined as the greatest detour distance from r to any other vertex within H. The detour radius $r_D(H)$ of H corresponds to the minimum detour eccentricity observed among its vertices, and the detour diameter $d_D(H)$ of H signifies the maximum detour eccentricity among its vertices. This detour concept finds broad application in addressing specific molecular challenges in theoretical chemistry [7].The graph parameters like radisu, diameter were extensively discussed in [9,10,11].

This paper explores a novel metric, referred to as circular distance, for measuring the distance between any two vertices in a graph H. This metric is derived by combining the sum of geodesic distance and detour distance within the graph H, and the research focuses on deriving various outcomes related to trees. The circular distance plays a crucial role in logistics management scenarios. When a delivery person intends to deliver goods to a new location and return to the initial position, they must choose the delivery points and input them into the system. The system then automatically generates a map, selecting a detour path that includes more delivery points and returning to the starting position by opting for a geodesic distance that involves fewer delivery points. The extension of this article can be extended in to Neutrosophic theory studied by the several authors. The distance is the major role for finding shortest distance using minimal spanning trees ref. [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, and 23]

2. Circular Distance

Definition 2.1. Within a connected graph H, considering any pair of vertices r and s, the circular distance between them is defined as the sum of detour and geodesic distances. This is symbolized as $cir(r,s)$. Alternatively $cir(r,s) = D(r,s) + d(r,s)$ when r is not equal to s, and $cir(r,s) = 0$ when $r = s$.

Example 2.2. Consider (5, 6) graph H, shown in the figure 1. In this graph, the geodesic distances and the detour distances, are shown in tables 1 and 2. From the tables, we write circular distances which are presented in table3.

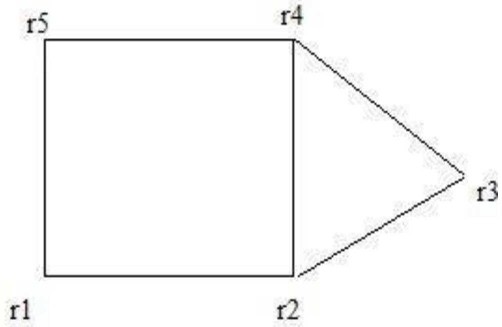

Figure 1. Graph H

$d(r_i, r_j)$	r_1	r_2	r_3	r_4	r_5
r_1	0	1	2	2	1
r_2	1	0	1	1	2
r_3	2	1	0	1	2
r_4	2	1	1	0	1
r_5	1	2	2	1	0

TABLE 1. Geodesic Distances of a Graph H

$D(r_i, r_j)$	r_1	r_2	r_3	r_4	r_5
r_1	0	4	4	3	4
r_2	4	0	4	3	3
r_3	4	4	0	4	4
r_4	3	3	4	0	4
r_5	4	3	4	4	0

TABLE 2. Detour Distances of a Graph H

Next, we calculate the circular distances between vertices of theabove graph. From above two tables,the geodesic distance $d(r_1, r_2) = 1$ and detour distance, $D(r_1, r_2) = 4$ between r_1 and r_2. Thus circular distance, $cir(r_1, r_2) = 5$ of graph H. Similarly, the circular distances between remaining vertices can be calculated and they are as follows.

$cir(r_i, r_j)$	r_1	r_2	r_3	r_4	r_5
r_1	0	5	6	5	5
r_2	5	0	5	4	5
r_3	6	5	0	5	6
r_4	5	4	5	0	5
r_5	5	5	6	5	0

TABLE 3. Circular Distances between vertices of Graph H

Next we prove an important result on circular distance.We use the fact that geodesic distance and detour distances are metrices

Theorem 2.3. *The circular distance is a metric on the set of all vertices of connected graph* H.

Proof: In a connected graph H, let the vertices be *r, s, t.* Suppose $cir(r,s) \geq 0$ and $cir(r,s) = 0$ if and only if $r = s$. Also we have, $cir(r,s) = cir(s,r)$, hence this distance is symmetric.

The geodesic distance and detour distance follows the triangular inequality. Next we have

$$\begin{aligned} cir(r,t) + cir(t,s) &= \{d(r,t) + D(r,t)\} + \{d(t,s) + D(t,s)\} \\ &= \{d(r,t) + d(t,s)\} + \{D(r,t) + D(t,s)\} \\ &\geq d(r,s) + D(r,s) \\ &\geq cir(r,s) \end{aligned}$$

Thus $cir(r,t) + cir(t,s) \geq cir(r,s)$. Hence the Theorem.

Next, we prove a relation between circular radius and circular diameter.

Theorem2.4.*For a connected graph* H*, the circular radius and the circular diameter satisfy the inequality* $r^C(H) \leq d^C(H) \leq 2r^C(H)$.

Proof: By definition, the lower bound is clearly true. Using triangular inequality upper bound can be proved.

Let x and y are two vertices in the vertex set of graph H, the circular distance between x and y is equal to circular diameter of H. Let z be the vertex of H such that $e^C(z) = r^C(H)$.

We have

$$\begin{aligned} d^C(H) &= cir(x,y) \\ &\leq cir(x,z) + cir(z,y) \\ &\leq e^C(z) + e^C(z) \\ &\leq 2e^C(z) \\ &\leq 2r^C(H) \end{aligned}$$

Therefore $d^C(H) \leq 2r^C(H)$. Hence $r^C(H) \leq d^C(H) \leq 2r^C(H)$.

In the next theorem, we prove that the above bounds are sharp.

Theorem 2.5. *There exists a class of connected graph* H_K , such that $d^C(H_K) = 2r^C(H_K)$.

Proof: If H be a path graph P_5 with r_1, r_2, r_3, r_4 and r_5 as the vertices. Define H_K to be the graph which is obtained by adding $k(\geq 2)$ pendent vertices to r_2 and r_4 .Let $s_1, s_2, \cdots, s_k$ be the vertices adjacent to r_2 and $t_1, t_2, \cdots, t_k$ be the vertices adjacent to r_4. For this graph, we can see that $e^C(r_i) = 2k-2, i=1$ and $5, e^C(r_3) = k-1,$ $e^C(s_i) = e^C(t_i) = 2k-2, \forall i = 1,2,\cdots,k$. Hence the circular radius $r^C(H_K) = k-1$ and the circular diameter $d^C(H_K) = 2k-2 = 2(k-1) = 2r^C(H_k)$. Therefore we have $d^C(H_K) = 2r^C(H_K)$.

Theorem 2.6. *There exists a class of connected graphs H such that $r^C(H) = d^C(H)$.*

Proof: In complete graph K_n, degree of each vertex is $n-1$. The circular distance between any two vertices is $1 + n - 1 = n$. Thus circular eccentricity of every vertexis n. Hence circular radius and circular diameter of K_n are equal and is equal to n.

Any non- trivial graph can be embeded as the center of a connected graph, as show in the next theorem. Given a non- trivial graph *H,* we can construct a connected graph *F* such *H* is the circularcenter of *F,* as shown below.

Theorem 2.7. *The vertex set of every graph H (with at least two vertices) is the vertex-to-vertex circular center of some connected graph F.*

Proof: Let *H* be a graph with *n* vertices.We show that *H* is the center of some connected graph *F* (i.e., vertex set of *H* is the center of *F*). First, if *H* is not complete, make *H* complete graph by adding all them is singe dges. Thenadd two new vertices *r* and *s* to *H* andjoin them to every vertex of *H* but not each other.Next we add two more vertices *k* and *l* and join *k* to *r* and *l* to *s.* The resulting graph *F* is as shown in figure 2. Then it is clear that $e^C(k) = e^C(l) = n+7.$ $e^C(r) = e^C(s) = n+5$ and $e^C(t) = n+4$ for every vertex *t* in *H.* Hence the vertexset of *H* is th ecenter of *F* with respect to circular distance.

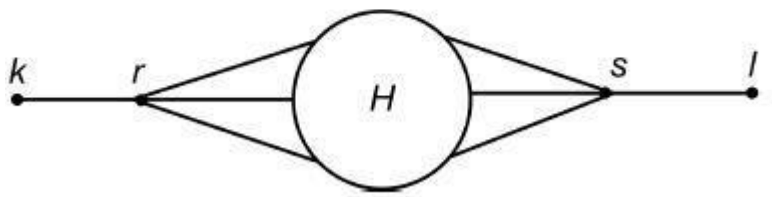

Figure 2. Construction of F for vertex-to-vertex circular distance

Next we prove that in any graph *H* there is a block which contains the circular center of *H.*

Theorem 2.8. *The circular center, $C^C(H)$, of any connected graph H lies with in a block of H*

Proof: Suppose the circular center, $C^C(H)$, of a connected graph *H* lies in more than one block. Then *H* contains a cut vertex *r* such that $H - r$ has at least two components H_1 and H_2 each of which contains a central vertex of *H.* Let s be a circular eccentric vertex of *r* and let *P* be a*s r* path of length $e^C(r)$. Then *P* contains novertex from at least one of H_1 and H_2, say H_1. Let t be a central vertex of H_1 and let P^1 be a $t-r$ geodesicin *H.* Then $e^C(t) \geq cir(t,r) + cir(r,s) \geq 1 + e^C(r)$. So t is not central vertex, which is a contradiction because $t \in C^C(H)$. Thus all central vertices must belie in a single block

3. Some Results on Trees

Theorem 3.1. *All eccentric vertices of a tree w.r.t. circular distance are pendent vertices only.*

Proof. In a tree there is a unique path between any two vertices and hence geodesic path is same as detour path. Thus the maximum distance from any vertex to anyother vertex occurs at end vertices only.Hence all circular eccentric vertices of a tree are pendent vertices.

Theorem 3.2. The center of a tree, with respect to circular distance, comprises either a solitary vertex or a pair of neighboring vertices.

Proof: The out come is straight forward for the trees $K_1$and K_2.

We will demonstrate that any tree T shares its center with the tree T'derived by eliminating all terminal vertices of T, Clearly, for each vertex r in T, only an end vertexcan serve as a circular eccentric vertex for r. Consequently, the circular eccentricity of each vertex in T'will be precisely 2 less than that of the corresponding vertex in *T, i.e., T and* T'share the same circular center.

By iteratively removing end vertices, we generate a sequence of trees, each possessing the same circularcenter as T. Given that T is finite, this process eventually yields a subtree of T, which is either K_1 or K_2. In either scenarios, the vertices in this ultimate tree constitute the circularcenter of T which thus comprises either a solitary vertex or a pair of neighboringvertices.

Theorem 3.3. *In any tree, the circular periphery exclusively comprises terminal vertices.*

Proof: Since the maximum circular distance from one vertex to othervertex in tree occurs at end vertices only the maximum circular eccentricity of a vertex is thecircular peripheral vertex, that occursonly at end vertices. Therefore, the circular periphery of tree consists of end vertices only.

4.Conclusion

This paper delves into an innovative metric termed circular distance, defined as the combination of geodesic distance and detour distance. Additionally, various properties of the circular distance have been examined, with a significant observation being that it serves as a metric for the entire set of vertices.

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